Comment on "Integrability of the Rabi Model"

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The Rabi model describes a two-level system coupled with a single bosonic mode [1]. Such a simplest interacting quantum model has possessed wide applications in many fields of physics [2-4]. Therefore, in order to understand thoroughly the physical properties of these systems, its exact solution, which has been a long-standing problem, is of special significance. However, Braak presented an analytical solution of the Rabi model seven years ago [5]. The energy spectrum consists of two parts, i. e. the regular and the exceptional spectrum. Such a spectrum structure is considerably strange. In this Comment, I point out that the Braak's analytical solution doesn't exhibit the true energy spectrum of the Rabi model due to the derivation error in solving the time-independent Schrodinger equation.

Braak started from the Rabi model

$$H_{sb} = \omega a^+ a + g\sigma_z(a^+ + a) + \Delta\sigma_x \tag{1}$$

in Ref. [5]. After taking the transformations $a \to \frac{\partial}{\partial z}$ and $a^+ \to z$, then the Hamiltonian (1) becomes

$$H_{sb} = \begin{pmatrix} \omega z\partial_z + g(z+\partial_z) & \Delta \\ \Delta & \omega z\partial_z - g(z+\partial_z) \end{pmatrix}. \tag{2}$$

Suppose that $(\phi_1\ \phi_2)^T$ is the two-component wave function of H_{sb}. Then one has a coupled system of the first-order differential equations

$$(z+g)\frac{d}{dz}\phi_1(z) + (gz-E)\phi_1(z) + \Delta\phi_2(z) = 0, \tag{3}$$
$$(z-g)\frac{d}{dz}\phi_2(z) - (gz+E)\phi_2(z) + \Delta\phi_1(z) = 0, \tag{4}$$

where $\omega = 1$ and E is the corresponding eigenvalue. Braak found that Eqs. (3) and (4) have the following solution

$$\phi_1(z) = e^{-gz}\sum_{n=0}^{\infty} K_n(x)\Delta\frac{(z+g)^n}{x-n}, \tag{5}$$
$$\phi_2(z) = e^{-gz}\sum_{n=0}^{\infty} K_n(x)(z+g)^n, \tag{6}$$

where $x = E + g^2$, E can take an arbitrary value, and the constants $K_n(x)$ satisfy the recursive relation (4) in Ref. [5]. Obviously, $\phi_1(z)$ and $\phi_2(z)$ are divergent at $z \to -\infty$. Therefore, this two-component solution $(\phi_1\ \phi_2)^T$ of H_{sb} is trivial and non-physical due to the divergence of the wave function and the undetermined eigenvalue.

In order to fix the eigenvalue E, Braak employed the unitary transformation

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ T & -T \end{pmatrix}, \tag{7}$$

where the operator T satisfies $T(f)(z) = f(-z)$. It is easy to get

$$U^+ H_{sb} U = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}. \tag{8}$$

Here, $H_\pm = \omega z\partial_z + g(z+\partial_z) \pm \Delta T$. Obviously, H_- can be obtained from H_+ by letting Δ be $-\Delta$.

The time-independent Schrodinger equation for H_+ with positive parity reads

$$z\frac{d}{dz}\psi(z) + g(\frac{d}{dz} + z)\psi(z) = E\psi(z) - \Delta\psi(-z), \tag{9}$$

which becomes

$$z\frac{d}{dz}\psi(-z) - g(\frac{d}{dz} + z)\psi(-z) = E\psi(-z) - \Delta\psi(z) \tag{10}$$

after manipulating T on two sides of Eq. (9). Here $\omega = 1$ and E is the eigenvalue of H_+ (or H_{sb}). It is obvious that $\psi(z)$ and $\psi(-z) = T(\psi)(z)$ in Eq. (9) or (10) are correlated due to the reflection operator T.

With the notation $\psi(z) = \phi_1(z)$ and $\psi(-z) = \phi_2(z)$, Eqs. (9) and (10) lead to Eqs. (3) and (4), respectively. Such a notation is the solution of the coupled equations (3) and (4) rather than the single equation (9) with the presence of T. I note that if and only if

$$G_+(x;z) = \phi_2(z) - T\phi_1(z) = \phi_2(z) - \phi_1(-z) \equiv 0 \tag{11}$$

for any z, this notation $\{\psi(z), \psi(-z)\}$ is the solution of Eq. (9).

Obviously, Braak treated $\psi(z)$ and $\psi(-z)$ as independent wave functions and neglected the condition (11). By requiring the wave function $\{\psi(z), \psi(-z)\}$ to be continuous at $z = 0$, i.e. $G_+(x; z = 0) = 0$, Braak obtained the eigenvalues E (see (3) and Fig. 1 in Ref. [5]). However, the constraint (11) does not hold for nonzero z under these eigenvalues E [see the expressions (5) and (6)]. So such a wave function $\{\psi(z), \psi(-z)\}$ with a cusp at $z = 0$ and the corresponding eigenvalue E are not these of H_+. Similarly, the solution for H_- with negative parity can be obtained by replacing Δ with $-\Delta$. Therefore, it is out of question that the energy spectrum shown in Figs. 2 and 3 in Ref. [5] is not that of the Rabi model (1) [6].

Braak also applied the similar technique to the generalized Hamiltonian by adding $\epsilon\sigma_z$ to (1) (i.e. (7) in Ref. [5]). However, the same derivation error occurs. The energy spectrum depicted in Fig. 4 in Ref. [5] is also incorrect [6].

Finally, I would like to mention that I have exactly solved the Rabi model by using the unitary transformation technique in the occupation number representation [7]. The complete energy spectrum is comprised of two double-fold degenerate sub-energy spectrum I and II.

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